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Hence $\beta=D$. Hence $A=-\beta^3$, $L=\beta^4+\beta(\delta+2b)^2$. Hence $L:A=-\beta:1$ gives $\delta\equiv b \pmod 3$. Then (8) becomes $d^2(d+\beta^2)\equiv 0$. Hence $d=0$ or $-\beta^2$, giving again algebras (2) and (14). Finally, let $\beta=0$. Then $A\equiv 0$, so that the Hessian must be identically zero. From C, K, we get

$$bdD(\delta+2b)+b^2D^3+d^3=0,\quad d^2D=d(\delta+2b)^2.$$

If $d\neq 0$, we eliminate $\delta+2b$ and get $(b^2D^3-d^3)^2=0$. Hence $(bDd^{-1})^3-b=0$, contrary to the irreducibility of (10). Hence $d=0$, $D=0$. Then $H=(\delta+2b)^3rst$, so that $\delta\equiv b$. The algebra is again the field algebra.

In summary, *the only algebras with the required properties are the obvious field-algebra and the remarkable non-field algebra* (14).

The University of Chicago, *September, 1906.*

THE DIVISION OF ANGLES INTO n EQUAL PARTS.

By J. SAMSONOFF, New York City.

Principle: If by the method given later one has constructed the geometrical loci of vertices of triangles with base XA, whose remote angle at the base is equal respectively to one, two, three, ..., $(n-1)$ times the angle at the vertex of the respective triangle, any angle CAY is geometrically divisible by 2, 3, ..., n.

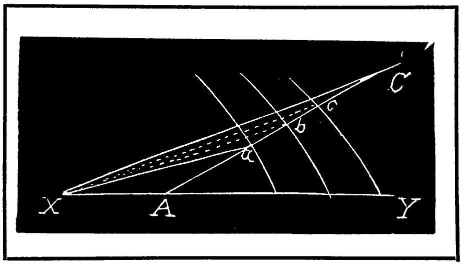

Fig. 1.

Given: Base XA and curves a, b, c, d, ..., the geometrical loci of vertices of triangles, where the remote

$$\angle aXA=\angle XaA,\quad \angle bXA=2\angle XbA,\quad \dots,\quad \angle eXA=(n-1)\angle XeA.$$

To prove: $\angle CAY$ is geometrically divisible into 2, 3, 4, ..., n equal parts.

Proof: (1) In $\triangle XaA$, $\angle aXA=\angle XaA$ (by hypothesis), $\angle CAY=\angle aXA+\angle XaA$ (the exterior angle of $\triangle aXA$). Therefore the measure of $\angle XaA$ must be equal to $\frac{1}{2}$ of the measure of $\angle CAY$.

(2) In $\triangle XbA$, $\angle bXA=2\angle XbA$ (by hypothesis), $\angle CAY=\angle bXA+\angle XbA$ (the exterior angle of $\triangle XbA$). Therefore the measure of $\angle XbA$ must be equal to $\frac{1}{3}$ of the measure of $\angle CAY$.

(n) In $\triangle XeA$, $\angle eXA=(n-1)\angle XeA$ (by hypothesis), $\angle CAY=\angle eXA+\angle XeA$ (the exterior angle of $\triangle eXA$). Therefore the measure of $\angle XeA$ must be equal to $1/n$ of the measure of $\angle CAY$.

Hence the problem to divide an angle into any number of equal parts depends upon the solution of the problem of drawing the respective curve.

Analysis of the method of drawing a curve as the geometrical locus of vertices of triangles where the remote angle at the base is equal to $(n-1)$ times the angle at the vertex.

Let XY be a given straight line from which (in relation to the points X and A) a n-sectorial curve (a) is drawn. In Fig. 2, $n=6$. Connect any point a_6 of the curve with the relative points X and A.

In $\triangle a_6XA$, $\angle a_6XA = 5(\angle Xa_6A)$, by hypothesis. Take point A as a center and with AX as a radius circumscribe a circle which intersects line a_6X in a_1. In $\triangle Aa_1a_6$, the exterior $\angle Xa_1A$ is equal to $\angle a_1Aa_6 + \angle Aa_6a_1$; but $\angle Xa_1A = \angle a_1XA$ by construction, and $\angle a_6XA = 5(\angle Xa_6A)$ by hypothesis. Hence $5(\angle Xa_6A) = \angle a_1Aa_6 + \angle Xa_6A$, or $\angle a_1Aa_6 = 4(\angle Xa_6A)$. For general n, $\measuredangle a_nXA = (n-1)(\measuredangle Xa_nA)$, $\measuredangle a_1Aa_n = (n-2)(\measuredangle Xa_nA)$.

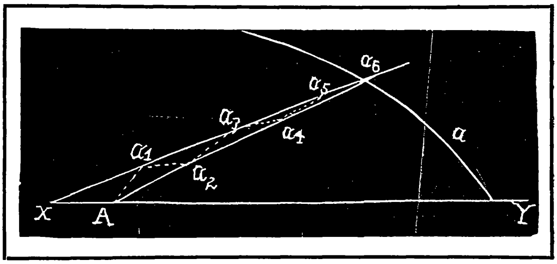

Fig. 2.

Take point a_1 as a center and with $a_1A = AX$ as a radius circumscribe a circle which will intersect line AC at a_2; then, reasoning as above, we will find that $\angle a_na_1a_2 = (n-3)(\angle Xa_nA)$. Repeating this construction, we will form new triangles, where one angle at the base will equal respectively $(n-4)$, $(n-5)$, ..., 4, 3, 2, and 1 times $\angle Xa_nA$. The last triangle will also be an isosceles triangle. Therefore, we must consider $\triangle Xa_6A$ as a sum of a series of isosceles triangles, where all sides are equal to XA.

The nearer to XY point a_6 on the curve is taken, the less is the difference between XY and Xa_6, the less the difference between AY and Aa_6; and when point a_6 coincides with Y, the broken line $Aa_1a_2\ldots a_6$ straightens itself and coincides with AY. Hence the relative points A and X for a required curve are: For a bisectorial curve, $XA = AY$; for a trisectorial curve, $XA = \frac{AY}{2}$; ...; for a n-sectorial curve,

$$XA = \frac{AY}{n-1}.$$

It remains to construct the instrument for drawing the curves.

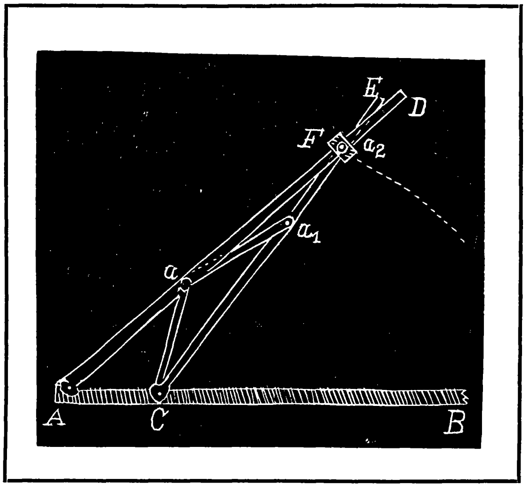

Fig. 3.

This instrument (Fig. 3) consists of a ruler AB supplied with pins so that when ruler AB is placed in coincidence with any straight line it will become fixed. At the points A and C the rulers AD and CE are attached in hinge-like fashion so that they may be freely movable. These rulers are also supplied with slits along their whole length. Also, at the point C hinges a folded ruler $Caa_1a_2a_3...$ consisting of parts Ca, aa_1, ..., each part being equal to AC. The rulers AD, CE, and the folded ruler $Caa_1a_2...$ are held together by a common handle F, which may be removed and replaced. This handle is supplied with a pencil for drawing.

When the ruler AB is placed on a straight line and the instrument is folded, the straight line is visible through the open slits of rulers CE and AD.

With this instrument we are able to solve the problem:

Draw a curve as the geometrical locus of vertices of triangles, the remote angle at the common base of which is $(n-1)$ times the angle at the vertex.

Solution: Apply instrument AB (Fig. 4) to the line XY. Bring down the ruler CE until its slit coincides with line XY. Unfold the folded ruler $Caa_1a_2a_3$ ... and take $(n-1)$ parts of it. Connect the odd points of the folded ruler with the ruler CE. Bring down the ruler AD until its slit coincides with line XY, and connect the even points of the folded ruler with the ruler AD. Now apply the handle F at the $(n-1)$st part of the folded ruler, fixing the rulers CE and AD. Move handle F (supplied with a pencil). The instrument will draw the required curve.

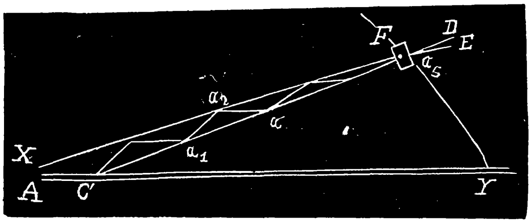

Fig. 4.

Let us prove that the drawn curve is the one required.

We have to prove that $\angle FCY$ is divisible geometrically into n equal parts.

Proof: $\triangle XFC$ is the sum of a series of $(n-1)$ isosceles triangles where the angles at the bases are successively equal to 1, 2, 3, 4, ..., $(n-1)$ times the angle at the vertex of $\triangle XFC$, by the construction of the instrument. [The points a, a_2, a_2, ..., of the folded ruler $Caa_1a_2a_3...$ will slide between the directrices AD and CE.] Therefore, in $\triangle XFC$, $\angle FXC=(n-1)(\angle XFC)$. Hence $\angle FCY$ is divisible into n equal parts.

But the triangles formed by moving handle F of the two connected directrices will include the series of $(n-1)$ isosceles triangles. Therefore the formed curve is the required one.